Homework 2: Integration, ODE

Exercise 2.1: Error Function

Consider the integral

$$E(x) = \int_0^x \mathrm{e}^{-t^2} \,\mathrm{d}t$$

- a) Write a program to calculate E(x) for values of x from 0 to 3 in steps of 0.1. Choose for yourself what method you will use for performing the integral and a suitable number of slices.
- b) When you are convinced your program is working, make a graph of E(x) as a function of x.

Note that there is no known way to perform this particular integral analytically, so numerical approaches are the only way forward.

Exercise 2.2: Heat capacity of a solid

Debye's theory of solids gives the heat capacity of a solid at temperature *T* to be

$$C_V = 9V\rho k_B \left(\frac{T}{\theta_D}\right)^3 \int_0^{\theta_D/T} \frac{x^4 e^x}{(e^x - 1)^2} \,\mathrm{d}x,$$

where *V* is the volume of the solid, ρ is the number density of atoms, k_B is Boltzmann's constant, and θ_D is the so-called *Debye temperature*, a property of solids that depends on their density and speed of sound.

- a) Write a Python function cv(T) that calculates C_V for a given value of the temperature, for a sample consisting of 1000 cubic centimeters of solid aluminum, which has a number density of $\rho = 6.022 \times 10^{28} \text{ m}^{-3}$ and a Debye temperature of $\theta_D = 428 \text{ K}$. Use Gaussian quadrature to evaluate the integral, with N = 50 sample points (*You may use example program gaussxw.py to generate the sample points and weights for any N and gaussint.py to see its use*).
- b) Use your function to make a graph of the heat capacity as a function of temperature from T = 5 K to T = 500 K.

Exercise 2.3: Harmonic and anharmonic oscillators

The simple harmonic oscillator arises in many physical problems, in mechanics, electricity and magnetism, and condensed matter physics, among other areas. Consider the standard oscillator equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x.$$

- a) Turn this second-order equation into two coupled first-order equations. Then write a program to solve them for the case $\omega = 1$ in the range from t = 0 to t = 50. A second-order equation requires two initial conditions, one on *x* and one on its derivative. For this problem use x = 1 and dx/dt = 0 as initial conditions. Have your program make a graph showing the value of *x* as a function of time.
- b) Now increase the amplitude of the oscillations by making the initial value of *x* bigger—say x = 2—and confirm that the period of the oscillations stays roughly the same.

c) Modify your program to solve for the motion of the anharmonic oscillator described by the equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x^3.$$

Again take $\omega = 1$ and initial conditions x = 1 and dx/dt = 0 and make a plot of the motion of the oscillator. Again increase the amplitude. You should observe that the oscillator oscillates faster at higher amplitudes. (You can try lower amplitudes too if you like, which should be slower.)

d) Modify your program so that instead of plotting x against t, it plots dx/dt against x, i.e., the "velocity" of the oscillator against its "position." Such a plot is called a *phase space* plot.

Exercise 2.4: Trajectory with air resistance

Many elementary mechanics problems deal with the physics of objects moving or flying through the air, but they almost always ignore friction and air resistance to make the equations solvable. If we're using a computer, however, we don't need solvable equations.

Consider, for instance, a spherical cannonball shot from a cannon standing on level ground. The air resistance on a moving sphere is a force in the opposite direction to the motion with magnitude

$$F = \frac{1}{2}\pi R^2 \rho C v^2,$$

where *R* is the sphere's radius, ρ is the density of air, *v* is the velocity, and *C* is the so-called *coefficient of drag* (a property of the shape of the moving object, in this case a sphere).

a) The equations of motion for the position (x, y) of the cannonball are

$$\ddot{x} = -\frac{\pi R^2 \rho C}{2m} \dot{x} \sqrt{\dot{x}^2 + \dot{y}^2}, \qquad \ddot{y} = -g - \frac{\pi R^2 \rho C}{2m} \dot{y} \sqrt{\dot{x}^2 + \dot{y}^2},$$

where *m* is the mass of the cannonball, *g* is the acceleration due to gravity, and \dot{x} and \ddot{x} are the first and second derivatives of *x* with respect to time.

- b) Change these two second-order equations into four first-order equations using the methods you have learned, then write a program that solves the equations for a cannonball of mass 1 kg and radius 8 cm, shot at 30° to the horizontal with initial velocity 100 ms⁻¹. The density of air is $\rho = 1.22$ kg m⁻³ and the coefficient of drag for a sphere is C = 0.47. Make a plot of the trajectory of the cannonball (i.e., a graph of *y* as a function of *x*).
- c) When one ignores air resistance, the distance traveled by a projectile does not depend on the mass of the projectile. In real life, however, mass certainly does make a difference. Use your program to estimate the total distance traveled (over horizontal ground) by the cannonball above, and then experiment with the program to determine whether the cannonball travels further if it is heavier or lighter. You could, for instance, plot a series of trajectories for cannonballs of different masses, or you could make a graph of distance traveled as a function of mass. Describe briefly what you discover.
- d) **Optional:** Calculate the total amount of heating of the cannon ball due to this friction. Assume all dissipated energy goes into heat.